ISRAEL JOURNAL OF MATHEMATICS 102 (1997), 61–99

ENTROPY OF TWIST INTERVAL MAPS*

BY

ALEXANDER BLOKH

Department of Mathematics, University of Alabama in Birmingham
University Station, Birmingham, AL 35294-2060, USA
e-mail: ablokh@math.uab.edu

AND

MICHAŁ MISIUREWICZ

Department of Mathematical Sciences, IUPUI
402 N. Blackford Street, Indianapolis, IN 46202-3216, USA
e-mail: mmisiure@math.iupui.edu

ABSTRACT

We investigate the recently introduced notion of rotation numbers for periodic orbits of interval maps. We identify twist orbits, that is those orbits that are the simplest ones with given rotation number. We estimate from below the topological entropy of a map having an orbit with given rotation number. Our estimates are sharp: there are unimodal maps where the equality holds. We also discuss what happens for maps with larger modality. In the Appendix we present a new approach to the problem of monotonicity of entropy in one-parameter families of unimodal maps.

* This work was partially done during the first author's visit to IUPUI (funded by a Faculty Research Grant from UAB Graduate School) and his visit to MSRI (the research at MSRI funded in part by NSF grant DMS-9022140) whose support the first author acknowledges with gratitude. The second author was partially supported by NSF grant DMS-9305899, and his gratitude is as great as that of the first author.

Received March 15, 1995 and in revised form November 20, 1995

Introduction

In the theory of dynamical systems, periodic orbits (cycles) play a very important role. The question of coexistence of various types of cycles for the same map admits particularly nice answers in dimension one. However, one has to decide what to consider the "type" of a cycle. For interval maps, two choices have been widely adopted. One of them is to look only at the period of a cycle. Then the results are very strong, namely the following Sharkovskiĭ theorem holds. To state it let us first introduce the **Sharkovskiĭ ordering** for positive integers:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2{\cdot}3 \succ 2{\cdot}5 \succ 2{\cdot}7 \succ \cdots 2^2{\cdot}3 \succ 2^2{\cdot}5 \succ 2^2{\cdot}7 \succ \cdots \succ 8 \succ 4 \succ 2 \succ 1$$

Denote by $\mathrm{Sh}(k)$ the set of all integers m such that $k \succeq m$, by $\mathrm{Sh}(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$, and by $P(f)$ the set of periods of cycles of a map f.

THEOREM 0.1 ([S]): *If $f\colon [0,1] \to [0,1]$ is continuous, $m \succ n$, and $m \in P(f)$ then $n \in P(f)$ and so there exists $k \in \mathbb{N} \cup \{2^\infty\}$ such that $P(f) = \mathrm{Sh}(k)$. Moreover, if $k \in \mathbb{N} \cup \{2^\infty\}$ then there exists a continuous map $f\colon [0,1] \to [0,1]$ such that $P(f) = \mathrm{Sh}(k)$.*

Unfortunately, the classification of cycles by period only is very coarse. The other choice, when one looks at the permutation determined by the cycle, has quite the opposite features. The classification is very fine, but the results are much weaker than for periods (see e.g. [ALM2]).

Recently a third possible choice has been discovered ([B1], see also [BK]). It gives a better classification than just by periods, and on the other hand, it admits a full description of possible sets of types. It is based on ideas from the theory of maps of a circle of degree one and employs rotation numbers. In this paper we will continue the investigation of connections between this approach and the second one (with permutations). Therefore, we have to start with a brief description of the approach via rotation numbers.

One can define rotation numbers in a variety of cases using the following approach (see, e.g. [MZ], [Z]). Let X be a compact metric space, let $f\colon X \to X$ be a continuous map, and let $\varphi\colon X \to \mathbb{R}$ be a bounded Borel function. Then let us call the number $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x))$ the **φ-rotation number** of x, whenever the limit exists. In particular, if x is periodic of period n, the φ-rotation number of x exists and is equal to $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x))$. With a proper choice of φ the set of all φ-rotation numbers of periodic points of f may contain a lot of information about the dynamics of f.

In the now classical case of rotation numbers of circle maps of degree 1 the function φ is the displacement for a lifting of the map. Therefore the sum $\sum_{i=0}^{n-1} \varphi(f^i x) = m$ taken along the orbit of an n-periodic point x is an integer. We call the pair $(m,n) \equiv rp(x)$ the **rotation pair** of x. Let us denote the set of all rotation pairs of periodic points of f by $\mathrm{RP}(f)$. Also, for real numbers $a \leq b$ set $M(a,b) = \{(p,q) \in \mathbb{Z} \times \mathbb{N}\colon p/q \in (a,b)\}$ (in particular $M(a,a) = \emptyset$). For $a \in \mathbb{R}$ and $l \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ let $Q(a,l)$ be empty if a is irrational or $l = 0$; otherwise let it be $\{(ks, ns)\colon s \in \mathrm{Sh}(l)\}$, where $a = k/n$ with k, n coprime (see [M1]).

THEOREM 0.2 ([M1]): *For a continuous circle map f of degree* 1 *there exist $a, b \in \mathbb{R}$ such that $a \leq b$, and $l, r \in \mathbb{N} \cup \{2^\infty\}$ such that $\mathrm{RP}(f) = M(a,b) \cup Q(a,l) \cup Q(b,r)$. Moreover, for every $a, b \in \mathbb{R}$ such that $a \leq b$ and $l, r \in \mathbb{N} \cup \{2^\infty\}$, there exists a continuous circle map f of degree* 1 *such that $\mathrm{RP}(f) = M(a,b) \cup Q(a,l) \cup Q(b,r)$.*

In the case of interval maps one can define rotation numbers in the following way. Let $f\colon [0,1] \to [0,1]$ be continuous, let $\mathrm{Per}(f)$ be its set of periodic points, and let $\mathrm{Fix}(f)$ be its set of fixed points. It is easy to see that if $\mathrm{Per}(f) = \mathrm{Fix}(f)$ then for any y the ω-limit set $\omega(y)$ is a set consisting of a fixed point of f. Assume that $\mathrm{Per}(f) \neq \mathrm{Fix}(f)$ and define a function χ as follows:

$$\chi(x) = \begin{cases} 1 & \text{if } f(x) < x, \\ \frac{1}{2} & \text{if } f(x) = x, \\ 0 & \text{if } f(x) > x. \end{cases}$$

In other words, χ is a slightly modified indicator function of the set $L = \{x\colon f(x) < x\}$. Then for any non-fixed periodic point y of period $p(y)$ the number $l(y) = \mathrm{card}\{\mathrm{orb}(y) \cap L\} = \sum_{i=0}^{p(y)-1} \chi(f^i y)$ is well-defined; the pair $rp(y) = (l(y), p(y))$ is called the **rotation pair** of y and the set of all rotation pairs of periodic non-fixed points of f is denoted by $\mathrm{RP}(f)$. Also, the χ-rotation number $\varrho_\chi(y) = \varrho(y) = l(y)/p(y)$ is simply called the **rotation number** of y.

THEOREM 0.3 ([B1]):

(1) *For a continuous interval map f with non-fixed periodic points there exist $0 \leq a \leq 1/2 \leq b \leq 1$ and $l, r \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ such that $\mathrm{RP}(f) = M(a,b) \cup Q(a,l) \cup Q(b,r)$, if $a < b = 1/2$ then $r = 3$, if $a = 1/2 < b$ then $l = 3$, if $a = b = 1/2$ then $r = l \neq 0$, if $a = 0$ then $l = 0$ and if $b = 1$ then $r = 0$.*

(2) *If a, b, l, r are numbers satisfying all the properties from statement (1) then there is a continuous interval map f such that*

$$\mathrm{RP}(f) = M(a,b) \cup Q(a,l) \cup Q(b,r).$$

Theorem 0.3 implies that the closure of the set of rotation numbers of all periodic points of f is an interval contained in $[0,1]$ and containing $1/2$. We call this interval the **rotation interval of** f and denote it by I_f.

A cycle is called **twist** if there is a map with this cycle and no other cycle of the same rotation number. This agrees with the terminology for circle maps (see e.g. [ALM2]). As usual, the **entropy of a cycle** is the minimal topological entropy of a map with this cycle. Section 2 is devoted mainly to the proof that twist cycles of entropy minimal among those with given rotation number, are unimodal.

In Section 3 we establish the connection between our rotation numbers and those introduced for unimodal maps by Gambaudo and Tresser in [GT]. We also derive a formula for the entropy of unimodal twist cycles. To state it, we need some definitions from [ALMT]. For $\mu \in [0,1], \mu \neq 1/2$, let $\lambda(\mu)$ be the unique root of the equation

$$\sum_{p/q \in (\mu; 1-\mu)} t^{-q} = 1$$

(in [ALMT] it was denoted $\beta_{\mu,1-\mu}$; also, we do not assume here that $\mu < 1-\mu$, so by the interval $(\mu; 1-\mu)$ we simply mean the interval with the endpoints μ and $1-\mu$). In the above equation we take the sum over all pairs of natural numbers p, q such that $p/q \in (\mu; 1-\mu)$. Note that in fact the number $\lambda(\mu)$ is defined for any $\mu \in [0,1], \mu \neq 1/2$, not only for rational ones. Also, $\lambda(\mu)$ is continuous as a function of μ.

We prove in Section 3 that the topological entropy of a unimodal twist cycle of rotation number μ is $\log \lambda(\mu)$. This, together with the results of Section 2, implies the following theorem.

THEOREM 3.9:

(1) *For any number $0 < \varrho < 1, \varrho \neq 1/2$, if the rotation interval of f contains ϱ then $h(f) \geq \log \lambda(\varrho)$. Thus, if a map f has a periodic point of rotation number ϱ then $h(f) \geq \log \lambda(\varrho)$.*

(2) *For every $\mu \in [0, 1/2)$ there is a continuous interval map with rotation interval $[\mu, 1/2]$ and topological entropy $\log \lambda(\mu)$.*

In Section 4, using a result of Bobok and Kuchta ([BK]), we describe a simple way of constructing all twist patterns (that is types of twist cycles, or in other words, their permutations). We also compute the number of twist patterns of a given rotation number.

In Section 5 we study some other possible connections between the rotation number and the entropy of twist cycles.

The equation for $\lambda(\mu)$ found in Section 3 is not very useful for concrete computations. Therefore, we give a more "computable" equation in the Appendix. Then we derive some estimates that shed light on the behavior of topological entropy in one-parameter families of unimodal maps.

1. Preliminaries

In this paper we provide estimates of the entropy of a map f given that it has a periodic point of rotation number ϱ. Furthermore, we estimate the entropy of a map f given that $\mu \in I_f$ for some $\mu \in [0,1]$ (we do not assume μ to be rational here). To explain our approach we have to introduce a few notions concerning so-called **combinatorial dynamics** (for this matter, we refer to [MN] and [ALM2], where one also can find much more extensive lists of references).

One can think of a **pattern** as a cyclic permutation of the set $1, 2, \dots, n$. According to [MN] we should rather use the term **cyclic pattern**, and according to [ALM2], **oriented pattern**. However, since we are dealing here only with them, we just say "pattern". If the restriction of an interval map f to its cycle P is conjugate to a pattern π by a strictly increasing map then P is a **representative** of π in f and f **exhibits** π on P ([MN]). A pattern π **forces** a pattern θ if every continuous interval map f which exhibits π also exhibits θ. Baldwin [Ba] showed that forcing is a partial ordering.

The infimum of the topological entropies of continuous maps exhibiting a pattern π is called the **entropy of the pattern** π. If $P = \{x_1 < x_2 < \cdots < x_n\}$ is a cycle of $f\colon [a,b] \to [a,b]$ then f is called P**-monotone** if it is constant on $[a, x_1]$, constant on $[x_n, b]$, and monotone on each one of the intervals $[x_i, x_{i+1}]$. If P is a representative of π in f and f is P-monotone then f exhibits only patterns forced by π and the topological entropy of f is equal to the entropy of π ([ALM2]).

Clearly one can talk about the **rotation pair** $\mathrm{rp}(\pi)$ and the **rotation number** $\varrho(\pi)$ of a pattern π of period larger than 1. The rotation number is invariant under a monotone semiconjugacy. More precisely, if a pattern π is semiconjugate to a pattern π' by a monotone (not necessarily strictly) increasing map then π' is said to be a **reduction** of π. Then the definition immediately implies that $\varrho(\pi) = \varrho(\pi')$. We call a pattern π a **twist pattern** or simply **twist** if it does not force other patterns of the same rotation number. Note that Theorem 0.3 implies that rotation pairs of twists consist of coprime numbers. In particular, the only twists of rotation number $1/2$ are of periods 1 and 2, and so from now on we consider only twists of rotation numbers distinct from $1/2$. Also we would like to make the following remark explaining to some extent the importance of twists. It is an immediate corollary to Theorem 0.3 and the fact that forcing is a partial ordering.

COROLLARY 1.1: *If a map f has a point of rotation number $\mu = p/q$ with p, q coprime, then f exhibits a twist pattern of rotation number μ. Moreover, any twist pattern of rotation number μ has rotation pair (p, q).*

In the particular case of unimodal patterns (by **unimodal** we mean patterns with a unique interior local extremum), a full characterization of twists was given in [B2]. For every rotation number $\varrho \neq 1/2$ there is a unique unimodal twist with this rotation number. We denote it by φ_ϱ. Before we describe the structure of φ_ϱ, note that it is enough to consider unimodal patterns whose only interior local extremum is a maximum. In terms of rotation numbers this translates into the observation that it is enough to deal with twists of rotation numbers smaller than $1/2$. Let us now state the following lemma.

LEMMA 1.2 ([B2]): *Let $p/q \neq 1/2$ be a rational number and let p, q be coprime. Then there is a unique unimodal twist $\pi = \varphi_{p/q}$ of rotation number p/q. If $p/q < 1/2$ then the twist π can be described as follows:*

(1) $\pi\colon \{1, \dots, q\} \to \{1, \dots, q\}$,
(2) $\pi(j) = j + p$ *for* $1 \leq j \leq q - 2p$,
(3) $\pi(j) = 2q - 2p + 1 - j$ *for* $q - 2p + 1 \leq j \leq q - p$,
(4) $\pi(j) = q + 1 - j$ *for* $q - p + 1 \leq j \leq q$.

If $p/q > 1/2$ then the twist $\varphi_{p/q}$ is defined similarly.

In fact the twist from Lemma 1.2 can be obtained as the result of the following procedure (see [B2]). Let $\theta\colon \{1, \dots, q\} \to \{1, \dots, q\}$ be defined as the shift by p to the right modulo q. In other words, let $\theta(j) = j + p$ if $1 \leq j \leq q - p$ and $\theta(j) = j + p - q$ if $q - p + 1 \leq j \leq q$. Let the permutation $\psi\colon \{1, \dots, q\} \to \{1, \dots, q\}$

be such that $\psi(j) = j$ if $1 \le j \le q-p$ and $\psi(j) = 2q-p+1-j$ otherwise (in other words, ψ is the identity on $1, \ldots, q-p$ and reverses orientation on $q-p+1, \ldots, q$). Now, $\pi = \psi^{-1} \circ \theta \circ \psi$. In other words, π is conjugate to θ via ψ.

Let us discuss the following question. Suppose that a rational number μ belongs to the rotation interval I_f of a map f. Does this imply that there is an f-periodic point with rotation number μ? By Theorem 0.3 the answer is affirmative if μ belongs to the interior of I_f. The case of endpoints is not so obvious in the general situation but for piecewise monotone maps the answer is still affirmative as is shown in [B3]. Namely, one can prove the following result.

PROPOSITION 1.3 ([B3]): *If f is piecewise monotone and $\mu \neq 0, 1$ belongs to I_f then there is a cycle of f of rotation number μ.*

Note that by Corollary 1.1 the periodic orbit from Proposition 1.3 may be chosen to represent a twist of rotation number μ.

Now we describe some tools developed in [B1]. Let $\mathcal{U}$ be the family of all self-mappings of the interval $[0, 1]$ with a unique fixed point, say, a. Let $f \in \mathcal{U}$. Then all points to the left of a are mapped to the right ($f(x) > x$) and all points to the right of a are mapped to the left ($f(x) < x$). We study a kind of symbolic dynamics for these maps. Let $I_0, I_1, \ldots$ be intervals having a as an endpoint (we call such intervals **admissible**) such that $f(I_j) \supset I_{j+1}$ for $0 \le j$. Then we say that $(I_0, I_1, \ldots)$ is an **admissible chain** of intervals (note that it need not be finite). If a finite admissible chain of intervals $(I_0, \ldots, I_{k-1})$ is such that $f(I_{k-1}) \supset I_0$ then we call $(I_0, \ldots, I_{k-1})$ an **admissible loop** of intervals. Clearly, in this case we have $k > 1$, since the image of an admissible interval cannot contain this interval.

Let φ be a function defined on the family of all admissible intervals such that $\varphi([b, a]) = 0$ if $b < a$ and $\varphi([a, c]) = 1$ if $a < c$. For an admissible loop $\bar{\alpha} = (I_0, \ldots, I_{k-1})$ let us call the pair of numbers (p, k), where $p = \sum_{j=0}^{k-1} \varphi(I_j)$, the **rotation pair** of $\bar{\alpha}$, and let us call the number $\varrho(\bar{\alpha}) = p/k$ the **rotation number** of $\bar{\alpha}$. We finish this series of definitions with the following one. A sequence $(y_1, \ldots, y_l)$ is called **non-repetitive** if it cannot be represented as several repetitions of a shorter sequence.

LEMMA 1.4 ([B1]): *Let $f \in \mathcal{U}$ and let $\bar{\alpha} = (I_0, \ldots, I_{k-1})$ be an admissible loop. Then there are the following possibilities.*

(1) *Assume that k is even, $\varphi(I_j) = 0$ if j is even, and $\varphi(I_j) = 1$ if j is odd.*

Then f has a periodic point of period 2.

(2) *Otherwise, there is a periodic point $x \in I_0$ such that $x \neq a$, $f^j(x) \in I_j$ $(0 \le j \le k-1)$, $f^k(x) = x$ and so $\varrho(x) = \varrho(\bar{\alpha})$. Moreover, if the sequence of numbers $(\varphi(I_0), \dots, \varphi(I_{k-1}))$ is non-repetitive then* $\mathrm{rp}(x) = \mathrm{rp}(\bar{\alpha})$.

The above lemma and some other tools are helpful in studying general properties of twists which we are about to describe. The following lemma was proven in [B1].

LEMMA 1.5 ([B1]): *A twist forces a unique fixed point.*

Let π be a pattern with $\mathrm{rp}(\pi) = (r, n)$ and let P be a representative of π in f. Assume that $f \in \mathcal{U}$ (i.e. f has a unique fixed point); by Lemma 1.5 this is the case if π is a twist and f is P-monotone. It has already been mentioned that the cases when $2r \le n$ and $2r \ge n$ are similar, so we assume that $2r \le n$ (and so $\varrho(\pi) = r/n \le 1/2$). Denote $A_0 = \{x \in P\colon x < a, f(x) < a\}, A_1 = \{x \in P\colon x < a, f(x) > a\}, A_2 = \{x \in P\colon a < x\}$. Also, for any cycle $X = \{x_0, x_1 = f(x_0), \dots, x_{k-1} = f^{k-1}(x_0)\}$ of f let us denote the admissible loop $([x_0, a], [x_1, a], \dots, [x_{k-1}, a])$ by $\bar{\alpha}(X)$.

LEMMA 1.6 ([B1]): *Let π be a twist. Then the following hold.*

(1) *$f(A_2)$ lies to the left of a.*
(2) *f is increasing on A_0 and decreasing on $A_1 \cup A_2$.*
(3) *$f^2(x) < x$ for all $x \in A_1$.*

When working with maps $f \in \mathcal{U}$ we will use terminology similar to that from [ALM1] and [M3]. Namely, let $f \in \mathcal{U}$, let a be the unique fixed point of f, and let P be a cycle of f. Then all points $x \in P$ such that x and $f(x)$ lie to the same side of a are called **green** and all other points of P are called **black**.

LEMMA 1.7: *Assume that π is a twist, P is a representative of π in f, and f is P-monotone. Then the rotation interval of f is $[\varrho(\pi), 1/2]$.*

Proof: By Lemma 1.6 a twist of rotation number less than $1/2$ forces only patterns of rotation numbers less than or equal to $1/2$. Hence, the rotation interval of f is $[\mu, 1/2]$ for some μ. If $\mu < \varrho(\pi)$ then by Theorem 0.3 there is a pattern γ of rotation number $m/n < \varrho(\pi)$ forced by π. By the same theorem γ forces a pattern of rotation number $\varrho(\pi)$. This contradicts the definition of a twist. ∎

Let us now turn to well known results allowing us to estimate (and in some cases even compute) the topological entropy of a map. We say that two non-degenerate closed intervals are **almost disjoint** if their intersection consists of at most one point. Now, if $f\colon [0,1] \to [0,1]$ is continuous and I_j, $j = 0, 1, \ldots, n-1$, are pairwise almost disjoint intervals then the **matrix associated with these intervals** is the $n \times n$ matrix $A = (a_{ij})_{i,j=0}^{i,j=n-1}$ such that $a_{ij} = 1$ if $f(I_i) \supset I_j$ and $a_{ij} = 0$ otherwise. In particular, if P is a representative of π in f and f is P-monotone, then the points of P divide the interval into pairwise almost disjoint subintervals. We forget about the intervals to the left of the leftmost point of P and to the right of the rightmost point of P. Then the corresponding associated matrix depends only on π, and is called the **matrix associated with** π. Finally, let us denote the spectral radius of a matrix B by $r(B)$.

LEMMA 1.8 ([BGMY]):

(1) *If $f\colon [0,1] \to [0,1]$ is continuous, $[0,1]$ is partitioned into n pairwise almost disjoint intervals $\{I_j\}_{j=0}^{n-1}$ and A is the corresponding associated matrix then $h(f) \geq \ln r(A)$.*
(2) *If π is a permutation and A is the matrix associated with π then the entropy of π is $\ln r(A)$.*

2. Twists of minimal entropy are unimodal

The aim of this section is to prove the following theorem.

THEOREM 2.1: *If π is a non-unimodal twist and θ is a unimodal twist of the same rotation number then $h(\pi) > h(\theta)$.*

As was explained in the introduction, this implies that the entropy of any map which contains μ in its rotation interval is larger than or equal to $\lambda(\mu)$. Throughout this section we consider a twist pattern π and its representative P in a P-monotone map $f\colon [0,1] \to [0,1]$. Then by Lemma 1.5, $f \in \mathcal{U}$. The unique fixed point of f will be denoted by a. The definition implies that all local extrema of f belong to P. In various lemmas additional assumptions about π will be made. We prove a few lemmas which lead to Theorem 2.1. Lemma 2.2 relies upon Lemma 1.6 and some results of [BGMY] stated in Section 1.

LEMMA 2.2: *If f is not unimodal then $h(f) \geq (1/2) \ln 3$.*

Proof: Assume that f is not unimodal. Then there are points $x_l < x_m < x_r$ of P such that $f(x_l) > f(x_m)$ and $f(x_r) > f(x_m)$. By Lemma 1.6 (2), at least one of the points x_m, x_r belongs to A_0, and therefore is green. Similarly, at least one of the points x_l, x_m belongs to $A_1 \cup A_2$, and therefore is black. Hence, P has a black point to the left of a green point. The rightmost point of $P \cap [0, a]$ is black and by Lemma 1.6 (1) all points of P to the right of it are also black. Therefore there is $k \geq 1$ and consecutive (in space) points $y_0 < y_1 < \cdots < y_k < y_{k+1}$ of P such that y_0 and y_{k+1} are black, $y_1, \ldots, y_k$ are green, and $y_{k+1} < a$.

We claim that $f^2(y_{k+1}) \leq y_0$. Suppose that this is false. Then $f^2(y_{k+1}) \geq y_1$. This implies that the set $A = [y_1, f(y_{k+1})] \cap P$ is invariant. Indeed, all green points from A are mapped to the left of a and therefore their images remain in A. If $\zeta \in A \cap [y_1, a]$ is a black point then $y_{k+1} \leq \zeta < a$, so by Lemma 1.6 (2), $a < f(\zeta) \leq f(y_{k+1})$, and hence $f(\zeta) \in A$. Finally, if $\zeta \in A \cap [a, f(y_{k+1})]$ then, again by Lemma 1.6 (2), $f(\zeta) \in [f^2(y_{k+1}), a] \subset [y_1, a]$, and thus $f(\zeta) \in A$. Hence A is an invariant proper subset of P (indeed, $y_0 \in P \setminus A$), which is impossible since P is a cycle. This proves our claim. Thus, there exists $z \in (a, f(y_{k+1})]$ such that $f(z) = y_0$. Note that by Lemma 1.6 (2), $f(y_0) > f(y_{k+1}) > z$.

By Lemma 1.8, in order to complete the proof it is enough to find three almost disjoint intervals I, J, K such that the f^2-image of each one of them contains $I \cup J \cup K$. Set $K = [y_{k+1}, a]$. The fact that y_k is a green point and y_{k+1} is a black point implies that there is a point $\xi \in (y_k, y_{k+1})$ such that $f(\xi) = a$. Set $J = [\xi, y_{k+1}]$ and $I = [y_0, \xi]$. Clearly, I, J, K are almost disjoint and $f(I) \cap f(J) \cap f(K) \supset [a, z]$. Hence $f^2(I), f^2(J), f^2(K)$ contain $I \cup J \cup K$. This completes the proof. ∎

Lemma 2.2 easily implies the following corollary.

COROLLARY 2.3: *If f is not unimodal and $1/3 \leq \varrho(\pi) < 1/2$ then $h(f) > h(\varphi_{\varrho(\pi)})$.*

Proof: By Lemma 1.8, $h(\varphi_{1/3}) = \ln \mu$, where $\mu = (1 + \sqrt{5})/2$, and $h(\varphi_{\varrho(\pi)}) \leq h(\varphi_{1/3}) = \ln \mu$. At the same time by Lemma 2.2, $h(f) \geq \ln \sqrt{3}$. Therefore the corollary follows from the fact that $\sqrt{3} > \mu$. ∎

To prove Theorem 2.1, it remains to consider twists π such that $\varrho(\pi) < 1/3$.

LEMMA 2.4: *Let π be a non-unimodal twist such that $\varrho(\pi) < 1/3$. Then $h(f) \geq \ln 2$.*

Proof: Assume that $h(f) < \ln 2$ and establish some properties of f that follow from the assumption.

STEP 1: If $x \in P$ then the following is impossible: $x < f^3(x) < f(x) < a < f^2(x)$.

Let $\bar{\alpha}$ be the admissible loop formed by the intervals joining a with the consecutive (in time) points of P. If $x < f^3(x) < f(x) < a < f^2(x)$ then we can consider the admissible loop $\bar{\beta}$ obtained by erasing the intervals $[x,a],[f(x),a],[f^2(x),a]$ from the loop $\bar{\alpha}$. If $\mathrm{rp}(P) = (p,q)$ then $\mathrm{rp}(\bar{\beta}) = (p-1, q-3)$. Now the fact that $p/q < 1/3$ implies that $\varrho(\bar{\beta}) = (p-1)/(q-3) < p/q$. By Lemma 1.4, the existence of the admissible loop $\bar{\beta}$ implies that there is an f-periodic point y with rotation number $\varrho(y) = \varrho(\bar{\beta}) < p/q$. Since f is P-monotone, π forces θ, where θ is the pattern of the orbit Q of y. By Theorem 0.3, a Q-monotone map has a cycle R with $\varrho(R) = p/q$. Therefore θ forces γ, where γ is the pattern of R. Since $\pi \neq \theta$ and forcing is a partial ordering, we have $\pi \neq \gamma$. This is a contradiction, since π is a twist and it forces γ.

STEP 2: The following is impossible: $f^3(x) \leq y < x < f(x) < a < f^2(x) \leq f(y)$. In particular, it is impossible that $f^3(x) < x < f(x) < a < f^2(x) \leq f^4(x)$.

Let $I = [y,x], J = [x,f(x)], K = [f(x),f^2(x)]$. Then $f(I) \supset K$, $f(J) \supset K$, $f(K) \supset I \cup J \cup K$. Moreover, by Lemma 1.8, $h(f) \geq \ln r(A)$, where A is the 3×3 matrix defined as follows: $a_{ij} = 0$ if both i and j are smaller than 3 and $a_{ij} = 1$ otherwise. The characteristic polynomial of A is $-\lambda^3+\lambda^2+2\lambda = -\lambda(\lambda+1)(\lambda-2)$, and thus $r(A) = 2$. Therefore $h(f) \geq \ln 2$, which contradicts the assumption that $h(f) < \ln 2$.

Let us now complete the proof by showing how the assumption $h(f) < \ln 2$ leads to the contradiction. Let c be the point at which f attains the global maximum. Let us move backwards along the preimages of c belonging to P until we first meet a green point lying to the left of a and to the right of c; the existence of such a point follows from the fact that f is not unimodal. Denote this green point by d. Then by the choice of d we have $d < f(d) < a < f^2(d)$. Let us show that $f^3(d) < c$. Indeed, by Lemma 1.6 (3), $f^3(d) < f(d)$. Now, $f^3(d) \in (d, f(d))$ is impossible by Step 1. Moreover, if $f^3(d) \in (c,d)$ then $f^3(d)$ must be a black point because of the choice of d. By Lemma 1.6 (2) this implies that $f^4(d) > f^2(d)$, so we have $f^3(d) < d < f(d) < a < f^2(d) < f^4(d)$, which is impossible by Step 2. Therefore $f^3(d) \leq c$. At the same time $f(c) > f^2(d)$, since f attains the global maximum at c. Thus we have $f^3(d) \le c < d < f(d) < a < f^2(d) < f(c)$, which is impossible by Step 2. The contradiction completes the proof. ∎

3. Unimodal maps and the formula for minimal entropy

In this section we will derive a formula for the minimal entropy of a pattern of a given rotation number $\mu < 1/2$. In fact, our result will be even stronger than that: it will be a formula for the minimal topological entropy of a continuous interval map with rotation interval containing $[\mu, 1/2]$, and it will work also for irrational μ's (see Introduction). Moreover, we show that for every $\mu < 1/2$ there is a continuous interval map with topological entropy given by this formula and rotation interval $[\mu, 1/2]$. We also prove that for unimodal maps the left endpoint of the rotation interval is closely related to the rotation interval bound introduced by Gambaudo and Tresser in [GT].

In this section we work mainly with unimodal maps. There are several ways of defining unimodal maps. We have to choose one of them. Let $I = [0, 1]$. We shall call a continuous map $f: I \to I$ **unimodal** if there exists $c \in (0, 1)$ such that $f(c) = 1$, $f(1) = 0$, and f is increasing on the interval $[0, c]$ and decreasing on $[c, 1]$. Suppose there is a fixed point $b \in [0, c]$. Then clearly the map has a 2-horseshoe (more precisely, $f[b, c] \supset [b, c] \cup [c, 1]$ and $f[c, 1] \supset [b, c] \cup [c, 1]$); at the same time it is clear that all the points to the left of b are attracted by fixed points, i.e. behave trivially in the dynamical sense. Therefore without loss of generality we may also assume that there are no fixed points in $(0, c)$. If $x \in I$ then sometimes there is another point in I with the same image under f as x. We shall denote this point by x'. Thus, we have $f(x') = f(x)$ and $x' \neq x$. A unimodal map f has a unique fixed point in $[c, 1]$. We shall denote this fixed point by a. It is well known that the point a' exists if and only if the topological entropy of f is larger than or equal to $(1/2) \log 2$. Otherwise the rotation interval of f is degenerate, so we will assume throughout this section that $h(f) \ge (1/2) \log 2$.

Clearly any periodic point of a unimodal map f has rotation number less than or equal to $1/2$. Therefore the rotation interval I_f of f is of the form $[\mu, 1/2]$. We will refer to μ as the **lower rotation number of** f and denote it by $\mathrm{LRN}(f)$. One can give another interpretation of the rotation interval of a map (see [B3], [B4] for an alternative definition and discussion). For unimodal maps (and in fact for a much larger class of maps) it is especially simple. Namely, for any point x let us consider the set of all limits of the averages $\frac{1}{n} \sum_{i=0}^{n-1} \chi(f^i(x))$. It is proven in [B3] that the union of all these limits over all points of the interval coincides with I_f.

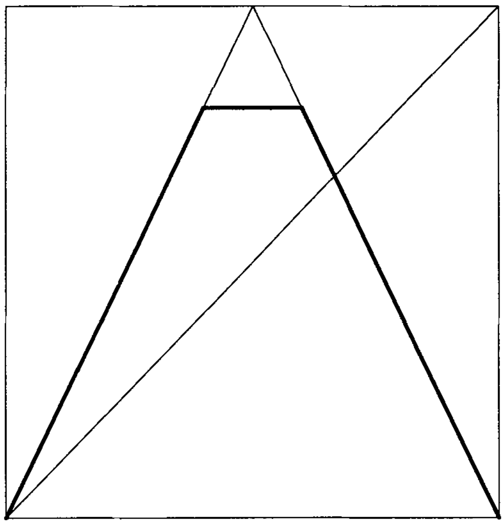

Figure 3.1. Truncated tent map

Let us look at the full tent map $T: I \to I$ given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

and at the truncated tent maps given by $T_\alpha(x) = \min(T(x), \alpha)$ (see [ALM2]; see also Figure 3.1). Although the maps T_α are not unimodal in our sense, they can serve as excellent models for unimodal maps. Therefore we will treat them as unimodal maps. If c is needed, we will choose a point from the plateau $[\alpha/2, 1 - \alpha/2]$ instead. Moreover, we will disregard the existence of the fixed point 0 if $\alpha < 1$. If we do this, then to find the lower rotation number of T_α it is enough to look at the rotation numbers of cycles of T_α and take their infimum. Set $\mathrm{LNR}(\alpha) = \mathrm{LNR}(T_\alpha)$. The rotation interval as a function of a truncation of a given (not necessarily unimodal) map is studied in [B2]. In the unimodal case the results of [B2] imply that the rotation interval $[\mathrm{LNR}(\alpha), 1/2]$ of T_α does not get smaller with the growth of α and depends on α continuously. This means that $\mathrm{LNR}(\alpha)$ is a non-increasing continuous function of α.

For any $\mu \in (0, 1/2)$ let $\beta(\mu)$ be the smallest number such that $\mathrm{LNR}(\beta(\mu)) = \mu$. In the next lemma we need the following observation connected with kneading invariants (see [CE]). It is well known that the kneading invariants of unimodal maps are linearly ordered. Suppose there are two different unimodal twist patterns π_1, π_2 of the same rotation number μ; denote their kneading invariants by ν_1 and ν_2 and suppose for definiteness that $\nu_1 > \nu_2$. If f is a unimodal map whose periodic critical point represents the pattern π_1 then by the well known properties of kneading invariants f has another periodic orbit representing the pattern π_2. This contradicts the fact that π_2 is a twist pattern. Therefore there is a unique twist unimodal pattern of a given rotation number μ. We will denote it by φ_μ.

LEMMA 3.1: *If μ is rational then every unimodal map with kneading invariant equal to that of $T_{\beta(\mu)}$ has periodic turning point c and the orbit of c has the unimodal twist pattern φ_μ of rotation number μ.*

Proof: Let us show that $T_{\beta(\mu)}$ has one of the endpoints of its critical interval periodic (denote this point by u), so that the orbit of u is a unimodal twist of rotation number μ. Indeed, otherwise there is a cycle Q of $T_{\beta(\mu)}$ of rotation number μ such that if $\nu = \max Q$ then $\nu < \beta(\mu)$. Consider the map T_ν. It has the cycle Q and so $\mathrm{LNR}(\nu) \leq \mu$, contradicting the definition of $\beta(\mu)$. This proves the lemma. ∎

We will call a unimodal map with kneading invariant equal to the kneading invariant of $T_{\beta(\mu)}$ for some μ, a **unimodal twist map**.

The following basic property of the lower rotation number for unimodal maps follows immediately from the kneading theory and the definition of LRN.

LEMMA 3.2: *For unimodal maps, the lower rotation number depends only on the kneading invariant. Moreover, this dependence is non-increasing.*

The following easy lemma was proven in [B1].

LEMMA 3.3 ([B1]): *A twist pattern is irreducible. In particular the kneading invariant of a unimodal twist map is indecomposable.*

We will be dealing with interval maps or circle maps that have finitely many discontinuities. Then a problem may arise, what should be the values of the map at the points of discontinuity. A wrong choice of those values may result in losing some rotation numbers. The simplest solution to this problem is to consider the one-sided limits at the points of discontinuity. Thus, for a map g, we will consider $(x, \lim_{y \nearrow x} g(y), \lim_{y \nearrow x} g^2(y), \dots)$ and $(x, \lim_{y \searrow x} g(y), \lim_{y \searrow x} g^2(y), \dots)$ as orbits. However, if $(x, g(x), g^2(x), \dots)$ is not one of those, we do not consider it as an orbit. Thus, we do not even have to define g at the points of discontinuity.

Let us fix a unimodal map f such that $\mathrm{LNR}(f) = \mu$. We will now discuss some results obtained in [B2], in order to explain how Lemma 1.2 describing unimodal twists can be proved. The main idea in [B2] is to change a unimodal map so that the results of [M4] are applicable. More precisely, using a discontinuous conjugacy and then a special lifting one can get a map of the real line into itself with special properties, called in [M4] an **old** map. For such maps the rotation numbers were introduced in [M4]. Our construction implies that the rotation numbers for the old map obtained in such a way, coincide with the rotation numbers for the original unimodal map. Therefore we can now rely upon [M4].

To realize this plan let us first define the discontinuous conjugacy $\sigma_c\colon I \to I$ as the identity on $[0, a]$ and the symmetry (with respect to $(a+1)/2$) on $[a, 1]$. Thus, $\sigma_c(x) = x$ if $0 \le x \le a$ and $\sigma_c(x) = a + 1 - x$ if $a \le x \le 1$. Now we define $\widetilde{f}\colon I \to I$ by $\widetilde{f} = \sigma_c \circ f \circ \sigma_c$ (see Figure 3.2).

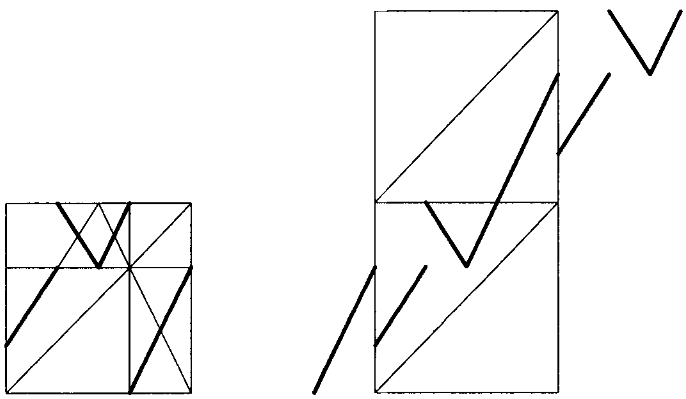

Figure 3.2. Functions $\widetilde{f}$ (left) and F (right)

We have to specify a lifting F of a map f. We do it by setting on $[0, 1)$

$$F(x) = \begin{cases} \widetilde{f}(x) & \text{if } 0 \le x \le a, \\ \widetilde{f}(x) + 1 & \text{if } a < x < 1, \end{cases}$$

and then as usual: if $x = k + y$ with $y \in [0, 1)$ then $F(x) = k + F(y)$. The map F we get in this way is an old map (see [M4]). Clearly, the usual rotation numbers for F are the same as our rotation numbers, except perhaps for the orbits passing through points of discontinuity. As we mentioned, in this case we consider one-sided limits. For them, arbitrarily long pieces of orbits of all maps under consideration are approximated by pieces of orbits of the same lengths that do not pass through any discontinuities. Therefore all rotation numbers for the two maps agree. Notice that with this approach a point of an orientation reversing cycle may be "cut in halves", so its period is twice as large as it should be. This does not affect its rotation number, so it does not harm us. On the other hand, the idea of cutting a point in halves is intriguing and may lead to the study of such interesting objects as intervals $(x, x]$ and $[x, x)$.

Thus, the rotation sets of f and F are the same. In particular, μ is the left endpoint of the rotation interval of F.

The following lemma was proved in [B2].

LEMMA 3.4 ([B2]): *If f is a unimodal twist map then the f-orbit of c misses the set $(a', c) \cup [a, 0')$. If μ is rational then the orbit of c has pattern described in Lemma 1.2.*

We are not going to reprove this lemma here. However, we will need some tools used in the proof, so we have to introduce them.

Assume that f is unimodal and let F be defined as above. The map F_l (defined by $F_l(x) = \inf_{y \geq x} F(y)$, see [M4], [ALM2]) has rotation number equal to the left endpoint of the rotation interval of F, that is to μ. This map coincides with F outside the open intervals where it is constant. In $[0, 1)$ they are the intervals $I = (a', c)$ and $J = \sigma_c(a, 0')$. Note that $F_l(I) = \{a\}$ and $F_l(a) = 1 \in J$. For f we get the compact f-invariant set A of all points missing intervals (a', c) and $(a, 0')$. Any of these points has rotation number μ and we have $c \in A$. Note that the F_l-orbit of c and the F-orbit of c (the right-hand limit) coincide.

Now let us return to the general situation. To every unimodal map f on $I = [0, 1]$ there corresponds a symmetric Lorenz-like map g on $[0, 2]$ (see [GT]). By symmetric Lorenz-like we mean that g is increasing on each one of the intervals $[0, 1]$ and $[1, 2]$, $\lim_{x \nearrow 1} g(x) = 2$, and if $g(1 + x) = 1 + y$ then $g(1 - x) = 1 - y$ (the graph is symmetric with respect to the point $(1, 1)$). We get the graph of g by reflecting the graph of f with respect to the line $x = 1$, then reflecting the resulting graph with respect to the line $y = 1$ (so when we fold the 2×2 square of paper in four and look through it, we see the same lines on all four quarters), and finally erasing all decreasing pieces of our picture (see [H], [GT]; see also Figure 3.3).

By the same method as for $\tilde{f}$, one can define rotation numbers for g (see [ALMT]). Strictly speaking, one should start by rescaling $[0,2]$ back to $[0,1]$. Then we look at the lifting G of g defined so that $G|_{(0,1/2)} = g|_{(0,1/2)}, G|_{(1/2,1)} = g|_{(1/2,1)} + 1, G|_{(k,k+1)} = G|_{(0,1)} + k$. The rotation interval of G may also be called the rotation interval of g. Since g is symmetric, its rotation interval is symmetric with respect to 1/2. It is easy to see that the G-orbit of 0 (the right-hand limit) and the G_l-orbit of 0 coincide if and only if the g-orbit of 0 misses $(2-0', 2]$.

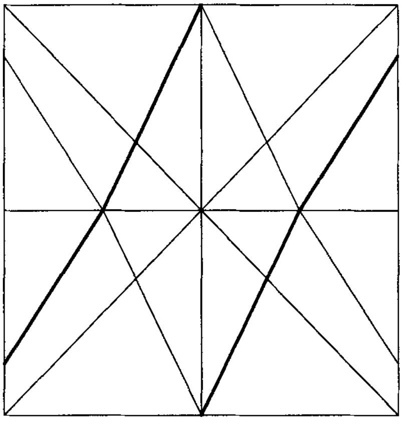

Figure 3.3. Lorenz-like map g corresponding to f

LEMMA 3.5: *Assume that f is a unimodal twist map. Then, if $f^n(0) \in [0,a'] \cup [c,a)$ then $g^n(0) = f^n(0)$, and if $f^n(0) \in [0',1]$ then $g^n(0) = 2 - f^n(0)$. In particular, the g-orbit of 0 misses $(2-0', 2]$.*

Proof: By Lemma 3.4, the f-orbit of 0 misses $(a', c) \cup [a, 0')$. We prove the first claim of the lemma by induction. For $n = 0$ it is clearly true. Assume that it is true for some n and prove it for $n+1$ replacing n.

Suppose first that $f^n(0) \in [0,a']$. By the inductive hypothesis, $g^n(0) = f^n(0) \in [0,a']$ and so $g^{n+1}(0) = g(f^n(0)) = f^{n+1}(0)$ (the maps f and g coincide on $[0,a']$). Since $f^{n+1}(0)$ cannot lie in (a',c), the point $g^{n+1}(0) = f^{n+1}(0)$ belongs to $[0,a] \setminus (a',c) = [0,a'] \cup [c,a]$. Hence, in this case the claim is true.

Suppose now that $f^n(0) \in [c,a)$. By the inductive hypothesis, $g^n(0) = f^n(0) \in [c,a)$ and so $g^{n+1}(0) = g(f^n(0)) = 2 - f^{n+1}(0)$ (we have $g(x) = 2 - f(x)$ for $x \in [c,a)$). Since $f^{n+1}(0)$ cannot lie in $(a,0')$, the point $g^{n+1}(0) = 2 - f^{n+1}(0)$ belongs to $[1, 2-a) \setminus (2-0', 2-a) = [1, 2-0']$. Hence, in this case the claim is also true.

Suppose at last that $f^n(0) \in [0',1]$. By the inductive hypothesis, $g^n(0) = 2 - f^n(0) \in [1,2]$ and so $g^{n+1}(0) = g(2 - f^n(0)) = f^{n+1}(0)$ (the maps $f(\cdot)$ and $g(2 - \cdot)$ coincide on $[0',1]$). Since $f^{n+1}(0)$ cannot lie in (a',c), the point $g^{n+1}(0) = f^{n+1}(0)$ belongs to $[0,f(0)] \setminus (a',c) \subset [0,a'] \cup [c,a]$. Hence, also in this case the claim is true.

We have proved the first claim of the lemma, and from this claim it follows immediately that $g^n(0)$ belongs to one of the intervals $[0,a']$, $[c,a)$, or $[1,2-0']$. Therefore the g-orbit of 0 misses $(2-0',2]$. ∎

PROPOSITION 3.6: *Assume that f is a unimodal twist map with lower rotation number μ. Then the rotation interval of g is $[\mu, 1-\mu]$.*

Proof: By symmetry, the rotation interval of g is $[\nu, 1-\nu]$, where ν is the rotation number of G_l. By Lemma 3.5, the G-orbit of 0 (the right-hand limit) and the G_l-orbit of 0 coincide. Therefore ν is the G-rotation number (right) of 0. The rotation numbers for G can be computed using g and the characteristic function of $[1,2]$, cohomologous to the displacement function. However, by Lemma 3.5, counting the number of times when $g^n(0)$ belongs to $[1,2]$ is equivalent to counting the number of times when $f^n(0)$ is in $(a,1]$. Therefore ν is equal to the f-rotation number of 0. Since $0 = f^2(c)$, ν is equal to the f-rotation number of c. This number is the same as the F-rotation number of c. Since the F_l-orbit of c and the F-orbit of c coincide, ν is equal to the rotation number of F_l, that is to μ. This completes the proof. ∎

Now we can prove the first of the main results of this section. In [GT], Gambaudo and Tresser introduced the **rotation interval bound** $\omega(f)$ for a unimodal map f. They worked with the upper map and an interval of length 2. Since we are working with the lower map and an interval of length 1, we prefer to use the rotation number of F_l, which is equal to $1 - (\omega(f)/2)$. We will call it the **GT-rotation number of** f.

THEOREM 3.7: *For unimodal maps the lower rotation number and the GT-rotation number coincide.*

Proof: By Lemma 3.2, the lower rotation number is a non-increasing function of the kneading invariant. From [GT] it can be easily deduced that the same is true for the GT-rotation number. Let f be a unimodal map. If its topological entropy is equal to $(1/2)\log 2$ then its lower rotation number is $1/2$, and it is easy to check that the same holds for its GT-rotation number. Therefore the same is true if the topological entropy of f is smaller.

The rest of the proof is simple because of the properties of both rotation numbers. In terms of kneading invariants, it is enough to show that if K is a kneading invariant then its GT-rotation number $\mathrm{GT}(K)$ equals its lower rotation number $\mathrm{LRN}(K)$. Let $\mathrm{LRN}(K) = \mu$ and let K' be the kneading invariant of the twist of rotation number μ. Also let $(K_1, K_2, \dots)$ be a sequence of kneading invariants of twists of rotation numbers $\mu_1 < \mu_2 < \cdots, \mu_i \to \mu$. By monotonicity of both rotation numbers we have $\mu = \mathrm{LRN}(K) = \mathrm{LRN}(K') = \mathrm{GT}(K') \geq \mathrm{GT}(K)$ and at the same time for any i the kneading invariant K_i is bigger than K (since $\mu_i < \mu$), so $\mu_i = \mathrm{LRN}(K_i) = \mathrm{GT}(K_i) \leq \mathrm{GT}(K)$. After taking the limit this implies that $\mu \leq \mathrm{GT}(K)$. Therefore $\mu = \mathrm{LNR}(K) = \mathrm{GT}(K)$. ∎

Now we are going to show that the formula from [ALMT] for the minimal entropy of the Lorenz-like maps with rotation interval $[\mu, 1-\mu]$ gives also the entropy of a unimodal twist map with lower rotation number μ. For this we need some definitions from [ALMT].

For $\mu \in [0, 1/2)$ let $\lambda(\mu)$ be the unique root of the equation

$$\sum_{\mu < p/q < 1-\mu} t^{-q} = 1$$

(in [ALMT] it was denoted $\beta_{\mu,1-\mu}$). In the above equation we take the sum over all pairs of natural numbers p, q such that $\mu < p/q < 1-\mu$. For $\gamma, \delta \in \mathbb{R}$ with $\gamma > 1$ we set $F_{\gamma,\delta}(x) = \gamma x + \delta$ for $x \in [0,1)$ and then extend $F_{\gamma,\delta}$ to $\mathbb{R}$ to an old map in the usual way. Note that we state Theorem 3.8 here in a bit more general form than in the Introduction, covering the cases of both rational and irrational μ.

THEOREM 3.8: *The topological entropy of a unimodal twist map with lower rotation number μ is $\log \lambda(\mu)$.*

Proof: Denote by S_λ the unimodal map of constant slope λ (see e.g.[ALM2]). By Lemma 3.3, any unimodal twist map has indecomposable kneading invariant. Then there exists a unimodal map of constant slope with this kneading invariant. Therefore for every $\mu \in [0, 1/2)$ there exists $\Lambda(\mu)$ such that $S_{\Lambda(\mu)}$ is a unimodal twist map with lower rotation number μ. Since the topological entropy of S_λ is $\log \lambda$, in order to prove the theorem it is enough to show that $\Lambda(\mu) = \lambda(\mu)$.

Fix $\mu \in (0,1/2)$. According to Proposition 15 of [ALMT], there exists δ such that the map $H_1 = F_{\lambda(\mu),\delta}$ has rotation interval $[\mu, 1-\mu]$. The map H_2 with graph symmetric to the graph of H_1 with respect to the point $(1/2, 1)$ has rotation interval symmetric to the rotation interval of H_1 with respect to $1/2$, that is also $[\mu, 1-\mu]$. The map $H_0 = (1/2)(H_1 + H_2)$ lies between H_1 and H_2 (i.e. either $H_1 \leq H_0 \leq H_2$ or $H_2 \leq H_0 \leq H_1$), and therefore its rotation interval is also $[\mu, 1-\mu]$. It is a map of constant slope $\lambda(\mu)$, so its entropy is $\log \lambda(\mu)$. On $(0,1)$ this map is affine and $H_0(1/2) = 1$. Since $\mu > 0$, H_0 has no fixed points and hence $\lambda(\mu) < 2$. Therefore H_0 is a lifting of the symmetric Lorenz-like map of constant slope $\lambda(\mu)$. This Lorenz-like map can be obtained from $S_{\lambda(\mu)}$ by the procedure described earlier in this section. By Theorem 3.7 the lower rotation number of $S_{\lambda(\mu)}$ is μ. This proves that $\lambda(\mu) \geq \Lambda(\mu)$.

On the other hand, by Theorem 3.7 the Lorenz-like map obtained from $S_{\Lambda(\mu)}$ has rotation interval $[\mu, 1-\mu]$ and topological entropy $\log \Lambda(\mu)$ (since it has constant slope $\Lambda(\mu)$). By Theorem C of [ALMT], this entropy is larger than or equal to $\log \lambda(\mu)$, and we get $\Lambda(\mu) \geq \lambda(\mu)$. This completes the proof if $\mu > 0$.

It is easy to see that $\lim_{\mu \searrow 0} \Lambda(\mu) = 2 = \Lambda(0)$. By a property of $\lambda(\cdot)$ proved in [ALMT] we have $\lim_{\mu \searrow 0} \lambda(\mu) = \lambda(0)$. Therefore we get also $\Lambda(0) = \lambda(0)$. ∎

Combining Theorem 3.8 with the results of Section 2 we obtain Theorem 3.9 which is the main result of this section.

THEOREM 3.9:

(1) *For any number $0 < \varrho < 1, \varrho \neq 1/2$, if the rotation interval of f contains ϱ then $h(f) \geq \log \lambda(\varrho)$. Thus, if a map f has a periodic point of rotation number ϱ then $h(f) \geq \log \lambda(\varrho)$.*

(2) *For every $\mu \in [0, 1/2)$ there is a continuous interval map with rotation interval $[\mu, 1/2]$ and topological entropy $\log \lambda(\mu)$.*

Proof: (1) We may assume that $\varrho < 1/2$. By Theorem 0.3, for any rational $\mu \in (\varrho, 1/2]$ there is an f-periodic point of rotation number μ and thus by Corollary 1.1 f exhibits a twist pattern, say, γ, of rotation number μ. Now, if θ is the unimodal twist of rotation number μ then Theorem 2.1 implies that $h(f) \geq h(\gamma) > h(\theta) = \log \lambda(\mu)$. Choosing μ arbitrarily close to ϱ and making use of the continuity of $\lambda(\mu)$ we get the required result.

(2) This follows immediately from Theorem 3.8. ∎

Remark 3.10: Two unimodal maps with the same topological entropy have the same lower rotation numbers. Therefore we may look at the lower rotation number as a function of entropy. In other words, look at the function φ such that $\varphi(\lambda)$ is the lower rotation number of S_λ. By Lemma 3.4 the set A of parameters λ for which S_λ is a unimodal twist map, is contained in the set B of parameters λ for which the S_λ-orbit of c misses (a', c). Clearly, the complement of B is open, so B is closed. By [BM], B has Lebesgue measure zero. Moreover, if an interval in the parameter space is disjoint from A then φ is constant on it. Therefore the function φ is locally constant almost everywhere.

4. Constructing twist patterns

In this section we will describe a simple algorithm allowing us to construct all twist patterns of a given rotation number. Following the terminology of [ALM1] and [M3], we will call the patterns satisfying the conditions from Lemma 1.6 **green patterns**, and their representatives **green cycles**. Thus, Lemma 1.6 states that every twist pattern is green. In [BK], where Lemma 1.6 is also proved, green cycles are called **parallel unicycles**.

In [BK] Bobok and Kuchta gave a simple algorithm for checking whether a green cycle is twist (they call twist cycles **X-minimal orbits**). To each green cycle (P, φ) they assigned a **code** $K\colon P \to \mathbb{Z}$ as follows. Let the rotation pair of P be $\mathrm{rp}(P) = (n, m+n)$. This means that there are m points of P to the left of the fixed point a and n to the right of a. For the leftmost element x of P we set $K(x) = 0$, and then we define by induction: $K(\varphi(y)) = K(y) + n$ if $y < a$ and $K(\varphi(y)) = K(y) - m$ if $y > a$. Notice that when we get back to x, n has been added m times and m has been subtracted n times, so we get again 0. A code is called **monotone** if it is increasing to the left of a and decreasing to the right of a.

For the sake of completeness we include our proof of the following theorem from [BK].

THEOREM 4.1 ([BK]): *A green cycle is twist if and only if its code is monotone.*

Before we prove this theorem, we prove a lemma which will be used also later.

LEMMA 4.2: *If m and n are coprime then codes for different points of P are different.*

Proof: Assume that m and n are coprime, and let $z \neq y$ be two points of P. We get $K(z)$ from $K(y)$ by adding i times n and subtracting j times m for some non-negative integers i, j such that $i \leq m$, $j \leq n$, and $0 < i + j < m + n$. If $K(z) = K(y)$ then $in = jm$. Since n and m are coprime, it follows that n divides j and m divides i, and hence $m + n \leq i + j$. This is impossible, and thus $K(z) \neq K(y)$. ∎

Proof of Theorem 4.1: Let P be a green cycle of rotation pair $(n, m+n)$ of a P-linear map f. Let a be the unique fixed point of f. We modify the code by dividing it by $m+n$, and denote the new code by L. That is, $L(x) = K(x)/(m+n)$ for $x \in P$. Then $L(f(x)) = L(x) + \mu - \chi(x)$, where $\mu = n/(m+n)$. We know that if P is twist then m and n are coprime. It is easy to check that if the code is monotone then also m and n are coprime (we can assume that $m \neq n$ and then there are no two distinct points with the same code). Therefore we may assume in the proof that m and n are coprime.

Suppose first that P is twist, but its code is not monotone. By Corollary 1.1, m and n are coprime. Since the code is not monotone, there are points $x, y \in P$ on the same side of a and such that x is farther than y from a but $L(x) \geq L(y)$. By Lemma 4.2 we get $L(x) > L(y)$. We have $x = f^k(y)$ for some $k \in \{1, 2, \ldots, m+n-1\}$. Look at the admissible loop $(I_0, I_1, \ldots, I_{k-1})$ of intervals (see Section 1), where I_i is the interval having a and $f^i(y)$ as endpoints. Then $L(x) - L(y) = k\mu - \sum_{i=0}^{k-1} \chi(f^i(y))$. Therefore $(1/k)\sum_{i=0}^{k-1} \chi(f^i(y)) < \mu$. This means that the rotation number of the cycle R corresponding to our admissible loop is smaller than μ. Therefore the pattern π of P forces the pattern π' of R. By Theorem 0.3, the R-linear map exhibits a cycle of rotation number μ, so π' forces a pattern π'' of rotation number μ. Since π and π' have different rotation numbers, they are different. Therefore $\pi'' \neq \pi$. Since π forces π'', this means that π is not twist, a contradiction. This proves that if P is twist then its code is monotone.

Now we assume that the code of P is monotone and show that P is twist. For this we use the $(P \cup \{a\})$-graph of f (see [ALM2]). For every $P \cup \{a\}$-basic interval I we denote by $v(I)$ its endpoint that is farther from a than the other one. The other endpoint of I has code larger than $L(v(I))$ or is equal to a. Because of this, and because of the monotonicity of the code, if I f-covers a $P \cup \{a\}$-basic interval J then $L(v(J)) \geq L(f(v(I)))$, and equality holds only if $v(J) = f(v(I))$. Let R be a cycle of f and let $(I_0, I_1, \ldots, I_{k-1})$ be the corresponding loop (this time not necessarily admissible, just a loop formed by arrows in a graph) in the $(P \cup \{a\})$-graph of f. If we set $I_k = I_0$ then we get

$$\sum_{i=0}^{k-1} L(v(I_i)) = \sum_{i=0}^{k-1} L(v(I_{i+1})) \geq \sum_{i=0}^{k-1} L(f(v(I_i))) \\ = \sum_{i=0}^{k-1} L(v(I_i)) + k\mu - \sum_{i=0}^{k-1} \chi(v(I_i)).$$

Moreover, the equality holds only if $v(I_{i+1}) = f(v(I_i))$ for each i. Since P is green, this condition is equivalent to our loop being the fundamental loop, that is to $R = P$. Thus, $(1/k)\sum_{i=0}^{k-1} \chi(f^i(y)) \geq \mu$, and this means that the rotation number of R is greater than or equal to μ. Moreover, the equality holds only if $R = P$. This proves that P is a twist. ∎

Note that the last part of the above proof shows also that if the code is monotone then μ is an endpoint of the rotation interval of f.

Now let us make an observation that will allow us to simplify the search for twist patterns. Namely, in the situation described above the code of P is monotone if and only if it is increasing to the left of a. Indeed, φ restricted to the points of P to the right of a is decreasing, and for each such point y we have $\varphi(y) < a$ and $K(\varphi(y)) = K(y) - m$. Thus, if K is increasing to the left of a, it has to be decreasing to the right of a. This means that there is no need to look to the right of a. Therefore, instead of looking at the green cycle (P, φ) we can look at the corresponding **green pseudocycle** (R, ψ) defined by $R = \{x \in P\colon x < a\}$ and

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) < a, \\ \varphi^2(x) & \text{if } \varphi(x) > a. \end{cases}$$

In fact, R is a cycle itself, but we will not think of it as a cycle of any map, but as a "short version" of P. Clearly, we can reconstruct P from R.

Note that if a point $x \in R$ is green then $\psi(x) > x$ and if it is black then $\psi(x) < x$. We have n black and $m - n$ green points. By Theorem 0.3, if P is twist then m and n are relatively prime, so if we are looking for twist patterns, we can make this assumption.

The code K restricted to R can be obtained very easily. Namely, we start from the leftmost point z of R and set $K(z) = 0$. Then we proceed by induction: if y is green then $K(\psi(y)) = K(y) + n$ and if y is black then $K(\psi(y)) = K(y) + n - m$.

We can look at our green pseudocycle in the following way. Let us start from some point x of R and go along the pseudocycle R: $x, \psi(x), \psi^2(x), \ldots$, until we get back to x. Note which point is black and which is green. The information we get in such a way can be represented as m points on a circle, some of them black, some of them green (in the same order as we met them when going along R). Of course, two pictures like that are equivalent if we get one of them from the other one via an orientation preserving homeomorphism of a circle. We will call their equivalence classes **circular models**. Obviously, if we start from different points in a pseudocycle we get equivalent pictures, or, in other words, the same circular model.

In such a way for every twist pattern of rotation number $n/(m+n)$ we get a corresponding circular model with n black and $m-n$ green points.

THEOREM 4.3: *Let m and n be relatively prime positive integers such that $m > n$. Then the correspondence between twist patterns of rotation number $n/(m+n)$ and circular models with n black and $m-n$ green points is one-to-one.*

Proof: We have to show that from a circular model we can reconstruct (in a unique way) a twist pattern. We have a set Q of m points on a circle, n of them black and $m-n$ green. Let us assign a code L to them. We start at some $x \in Q$, set $L(x) = 0$ and then move along the circle. If z follows y in our journey, then we set $L(z) = L(y)+n$ if y is green and $L(z) = L(y)+n-m$ if y is black (exactly as for a green pseudocycle). Since there are n black and $m-n$ green points, when we get back to x we get again $L(x) = 0$. By Lemma 4.2, if $z \neq y$ are points of Q then $L(z) \neq L(y)$. Let q be the smallest of the numbers $L(y)$, $y \in Q$. Set $K(y) = L(y) - q$ for all $y \in Q$. Now transfer the points of Q to an interval in such a way that the code K is increasing. Since m and n are relatively prime, the codes for different points are different, so we can do it (in a unique way). It is easy to see that in such a way we get a green pseudocycle with an increasing code. As we noticed earlier, from this pseudocycle we can reconstruct a twist cycle. When we start from a twist cycle, get its circular model, and get back to a cycle using the procedure described above then we end up with a cycle of the same pattern as the one we started with. This completes the proof. ∎

The procedure described in the proof of Theorem 4.3 allows us to produce easily all twist patterns of a given rotation number. In particular, we can compute how many of them there are.

PROPOSITION 4.4: *Let m and n be relatively prime positive integers such that $m > n$. Then there are $\frac{1}{m}\binom{m}{n}$ twist patterns of rotation number $n/(m+n)$.*

Proof: By Theorem 4.3 it is enough to show that there are $\frac{1}{m}\binom{m}{n}$ circular models with n black and $m-n$ green points. We can think of those points as placed at the m-th roots of 1 (if the circle is the unit circle in the complex plane). There are $\binom{m}{n}$ choices for the positions of black points. If we rotate the circle by the angle $2k\pi/m$ ($k \in \{1, 2, \ldots, m-1\}$) then we get the same circular model. Thus the number of circular models with n black and $m-n$ green points is $\frac{1}{m}\binom{m}{n}$, provided no rotation as above maps all the black points to black ones. Suppose that it does. The orbit of any black point for this rotation has period $m/\gcd(m,k)$ (where $\gcd(\cdot,\cdot)$ denotes the greatest common divisor) and the set of all black points is the union of j such orbits for some j. Thus, $n = jm/\gcd(m,k)$. The number $i = m/\gcd(m,k)$ is a divisor of m, and $i > 1$ since $k < m$. Therefore m and n have a common divisor $i > 1$, a contradiction. This completes the proof. ∎

5. Rotation number and modality vs. entropy

We know already that for every rational number from $(0, 1/2)$ there is a twist pattern with this rotation number and entropy smaller than $\log 2$. Namely, this is a unimodal pattern. However, we can ask how the entropy of twist patterns behaves when the modality increases. We will show that various behaviors are possible. The entropy may stay bounded, but it also may increase to infinity.

We start by looking for twist cycles with large entropy. In fact, modality cannot be small if the entropy is large, but we will include the estimate for modality because we can get it easily.

Let (R, ψ) be a green pseudocycle. A **switch** will be an R-basic interval with endpoints of different color (let us recall that an interval $[x, y]$ is R-**basic** if $x, y \in R$ and there are no points of R in (x, y); see e.g. [ALM2]).

LEMMA 5.1: *Let (P, φ) be a green cycle and let (R, ψ) be the corresponding green pseudocycle. Then for every black point $x \in R$ we have $h(P) \geq \frac{1}{2}\log s(x)$, where $s(x)$ is the number of switches between $\psi(x)$ and x.*

Proof: Let f be the P-linear map and let a be the fixed point of f. Let x be a black point of R. For each switch J contained in $[\psi(x), x] = [f^2(x), x]$ we have $[a, f(x)] \subset f(J)$ and $[f^2(x), x] \subset f([a, f(x)])$. Therefore $J_1 \subset f^2(J_2)$ for all switches J_1, J_2 between $\psi(x)$ and x. Thus, there is an $s(x)$-horseshoe for f^2, and hence $h(P) = h(f) \geq \frac{1}{2} \log s(x)$. ∎

Let P be a green cycle. Since the leftmost point of P is green (we assume that the rotation number of P is less than $1/2$), and to the right of the fixed point there are only black points, the modality of P is equal to $2l - 1$, where l is the number of local maxima. The local maxima occur exactly at the leftmost points of the blocks of black points. Therefore l is also equal to the number of blocks of black points in P. The point of P immediately to the left of the fixed point is also black, so l is also equal to the number of blocks of black points in the corresponding green pseudocycle (and to the number of blocks of green points, since those numbers are equal).

Notice also that the switches separate blocks of black and green points. The leftmost point of a green pseudocycle is green and the rightmost one is black. Therefore the number of switches in a green pseudocycle is $2l - 1$, exactly the same as the modality of the corresponding green cycle.

PROPOSITION 5.2: *Let m and n be relatively prime positive integers such that $m - n$ is odd and $m > 3n$. Then there exist twist patterns of rotation numbers $2n/(m + 3n)$ and $(m - n)/(2m)$, both with modality $2n + 1$ and entropy larger than or equal to $\frac{1}{2} \log 2n$.*

Proof: Let us look at the green pseudocycle (R, ψ) defined as follows. We set $R = \{2i: i = 0, 1, \dots, m-1\} \cup \{m - n + 2i: i = 0, 1, \dots, n - 1\}$. The points $2i$ for $i = 0, 1, \dots, m - n - 1$ are green and we set for them $\psi(2i) = 2i + 2n$. The rest of the points of R are black and for a black point j we set $\psi(j) = j - (m - n)$. Thus, there are $m - n$ green points and $2n$ black points. Moreover, if K is the code for R then $K(j) = j$ for every $j \in R$. Hence, the code for R is monotone, so the corresponding green cycle P is twist. The only thing that is not immediately clear is that R is a cycle. To check it, observe first that ψ is one-to-one. Then note that $m-n$ and $2n$ are coprime, so $(m-n)k = 2ns$ is possible only if $k = 2nt$ and $s = (m - n)t$ for some t. Thus the pair $k = 2n, s = m - n$ is the one with the smallest sum for which $(m - n)k = 2ns$ holds. This sum is $m + n$, so for any $j \in R$ all the numbers $j, \psi(j), \dots, \psi^{m+n-1}(j)$ are distinct. This proves that (R, ψ) is a cycle, so our construction makes sense.

There are $m + n$ points in R, $2n$ of them black. Therefore the rotation number of P is $2n/(m + 3n)$.

Since $m > 3n$, we have $2(m-n) > m-n+2(n-1)+1$, i.e. there is at least one green point between the black point $m-n+2(n-1)$ and the leftmost black point $2(m-n)$ of the last block of black points. Therefore there are $n+1$ blocks of black points in R (n of them of length 1). Hence, the modality of P is $2n+1$. We have $\psi(2(m-n)) = m-n$ and there are $2n$ switches from $m-n$ to $2(m-n)$. Thus, by Lemma 5.1, the entropy of P is larger than or equal to $\frac{1}{2}\log 2n$.

Let us now look at the green pseudocycle R' symmetric to R. By "symmetric" we mean that we get R' from R by reversing the orientation of the real line and switching black and green colors. Thus, for instance the leftmost point of R will be the rightmost one of R', and green points of R will be black points of R'. Suppose that x was green in R. When moving to $\psi(x)$, we added $2n$ (that is the number of black points in R) to the code. In R', the point x is black, and when we are moving to $\psi(x)$, we subtract $2n$ (that is the number of green points in R') from the code. Similar things happen if x was black in R. Therefore the code for R' can be obtained by subtracting the code for R from $2m-2$. Thus, the code for R' is increasing, so the corresponding green cycle P' is twist.

The modality of P' is equal to the number of switches in R', which is the same as the number of switches in R, that is $2n+1$. There are $m+n$ points in R', $m-n$ of them black. Therefore the rotation number of P is $(m-n)/(2m)$.

We have (in R) $\psi(m-n-1) = m+n-1$, and since $m+n-1 = (m-n) + 2(n-1)+1$, there are $2n$ switches between $m-n-1$ and $m+n-1$. Thus, applying Lemma 5.1 to R', we get $h(P') \geq \frac{1}{2}\log 2n$. ∎

It is not immediately clear which rational numbers we get as rotation numbers in Proposition 5.2. We can see this from the following lemma.

LEMMA 5.3: *Let p and q be relatively prime positive integers.*

(a) *Assume that $0 < p/q < 1/3$ and p is even. Then there exist relatively prime positive integers m and n such that $m-n$ is odd, $m > 3n$, and $p = 2n$, $q = m+3n$.*

(b) *Assume that $1/3 < p/q < 1/2$ and q is even. Then there exist relatively prime positive integers m and n such that $m-n$ is odd, $m > 3n$, and $p = m-n$, $q = 2m$.*

Proof: Assume that $0 < p/q < 1/3$ and p is even. Set $n = p/2$ and $m = q-3n$. Then $p = 2n$ and $q = m+3n$. Since p is even, q is odd, and therefore $m-n = q-4n$ is odd. If m and n have a common divisor $i > 1$ then i divides $p = 2n$ and

$q = m + 3n$. Since p and q are relatively prime, this is impossible, and therefore m and n are relatively prime. This proves (a).

Assume now that $1/3 < p/q < 1/2$ and q is even. Set $m = q/2$ and $n = m - p$. Then $p = m - n$ and $q = 2m$. Since q is even, $m - n = p$ is odd. If m and n have a common divisor $i > 1$ then i divides $q = 2m$ and $p = m - n$. Since p and q are relatively prime, this is impossible, and therefore m and n are relatively prime. This proves (b). ∎

From Proposition 5.2 and Lemma 5.3 we get immediately the following theorem.

THEOREM 5.4: *For every $\varrho \in [0, 1/2]$ there exists a sequence of twist patterns with rotation numbers converging to ϱ and entropies increasing to infinity.*

Now we want to find twist patterns with large modality and relatively small entropy. Let m and n be relatively prime positive integers such that $m > n$. Let us look at the circular model with n black and $m - n$ green points and such that all black points are grouped in one block (and consequently, all green points are grouped in one block). Since there are 2 blocks in this model, we will call the corresponding twist pattern a **2B pattern of type** (m, n). According to the usual rules, we will call a representative of a 2B pattern of type (m, n) a **2B cycle of type** (m, n).

By looking at the circular model we are working with, we see that in the corresponding green pseudocycle R each green point is mapped to the next green point to the right, and the last one is mapped to the rightmost point of R (which is black). Similarly, each black point is mapped to the next black point to the left, and the last one is mapped to the leftmost point of R (which is green). It is also easy to see that the black points will have codes $i(m - n)$, $i = 1, 2, \dots, n$ and the green points will have codes jn, $j = 0, 1, \dots, m - n - 1$.

LEMMA 5.5: *A 2B pattern of type (m, n) has rotation number $n/(m + n)$ and modality $2\min(n, m - n) - 1$.*

Proof: Since there are m points in the corresponding green pseudocycle R, and n of them are black, the rotation number of our cycle is $n/(m + n)$.

Let us compute the number of blocks of black or green points in R. If $n < m - n$ then between two consecutive multiples of n there can be at most one integer divisible by $m - n$, so all blocks of black points have length 1. Similarly, if

$m - n < n$ then all blocks of green points have length 1. Therefore the modality of the corresponding pattern is $2\min(n, m-n) - 1$. ∎

Now we want to estimate the entropy of a 2B pattern.

PROPOSITION 5.6: *The entropy of any* 2B *pattern is smaller than or equal to* $\log 10$.

Proof: Let P be a 2B cycle of type (m, n) and let f be the P-linear map. Look at the $(P \cup \{a\})$-graph G of f (see e.g. [ALM2]), where a is the fixed point. Call a $(P \cup \{a\})$-basic interval **green** if both of its endpoints are green, and **black** otherwise.

Because of the way the green points are mapped, each green interval f-covers only some basic interval(s) to the right of itself (but to the left of a). Moreover, if $n < m - n$ then all blocks of black points have length 1, so a green interval f-covers only the next basic interval to the right, or two of them, if they are black.

With the black intervals, it is more complicated. If a black interval J f-covers some basic intervals to the left of a, these intervals are to the right of J. Moreover, it f-covers some basic intervals to the right of a, and those basic intervals f-cover in turn some basic intervals to the left of a. Those basic intervals to the left of a lie to the right of the image of one of the endpoints of J under f^2. Notice that this image is the point where this endpoint is mapped for the corresponding green pseudocycle.

Now we start to simplify the picture by changing the graph. We do it in such a way that the entropy does not decrease. Look at the generalized oriented graph H constructed in the following way ("generalized" means that there may be more than one arrow from a vertex to a vertex). The vertices of H are those vertices of G that correspond to the basic intervals to the left of a. The arrows of G between those vertices are also the arrows of H. Moreover, we add an arrow in H from a vertex A to a vertex B if there is a vertex C of G that is not a vertex of H and there are arrows from A to C and from C to B in G. Since f maps the part of the interval to the right of a to the part of the interval to the left of a in a monotone way, for every pair of vertices A and B of H there is at most one intermediate C as above. Therefore we add at most one additional arrow from A to B, and hence in H there are at most two arrows from A to B.

The entropy of a Markov graph can be computed as the upper limit of the

n-th root of the number of loops of length at most n (see e.g. [ALM2]). To every loop in G there corresponds a loop in H of the same or smaller length, and this correspondence is one-to-one. Therefore the entropy of H is larger than or equal to the entropy of G.

Let us denote the vertices of H (remember that they are basic intervals) by A_i with $i < j$ if the basic interval A_i is to the left to the basic interval A_j. We shall use the information about the f-covering that we already have in two different ways, depending on whether $m - n < n$ or $n < m - n$.

If $m - n < n$ then the blocks of green points have length 1. Therefore in the corresponding green pseudopattern the black points are mapped at most by 2 to the left. Hence, from a vertex A_i of H there are no arrows to A_j with $j < i - 2$. Moreover, if $j \le i$ then there is at most one arrow from A_i to A_j. Thus, we replace H by a new generalized oriented graph K with the same vertices and no arrows from A_i to A_j if $j < i - 2$, one arrow from A_i to A_j if $i - 2 \le j \le i$, and two arrows from A_i to A_j if $j > i$.

Clearly, there is some n such that there are paths of length n in K from every vertex to every vertex. This means that the n-th power of the transition matrix M of K is strictly positive. From the Perron–Frobenius theory of positive matrices we know that for every positive vector v the growth rate of $\|M^n v\|$ as n goes to infinity is equal to the spectral radius of M. Therefore if we find a positive vector v and a number α such that $Mv \le \alpha v$ then we will know that the spectral radius of M is smaller than or equal to α.

To specify the vector v we have to specify its component v_i for every vertex A_i of K. Then the i-th component of the vector Mv will be equal to the sum of v_j over all arrows from A_i to A_j and all vertices A_j of K. Let us set $v_i = 2^{N-i}$, where N is the number of vertices of K. Then the i-th component u_i of Mv is equal to $2^{N-(i-2)} + 2^{N-(i-1)} + 2^{N-i} + 2(2^{N-(i+1)} + \cdots + 2^{N-N})$ (one or two first summands may be not there, if $i = 2$ or $i = 1$). Hence,

$$u_i \le 2^{N-(i-2)}+2^{N-(i-1)}+2^{N-i}+2\sum_{j=1}^{\infty} 2^{N-i-j} = (4+2+1+2)2^{N-i} = 9v_i \le 10v_i.$$

Therefore $h(P) = h(G) \le h(H) \le h(K) \le \log 10$.

If $n < m - n$ then the blocks of black points have length 1. Therefore from a green vertex A_i of H there emerges only one arrow if the next vertex to the right is green (to this vertex), or only two arrows if the next two vertices to the right are black (to those vertices). There is also a special case – the rightmost green vertex of H, and there is only one arrow emerging from it. This arrow goes to the next vertex to the right, which is in this case black. The black vertices of H come in pairs of adjacent vertices. From each element of such a pair there emerge at most two arrows to each vertex to the right of this pair, at most one arrow to the vertices of the next pair to the left (the leftmost vertex if there are no black vertices to the left of our pair) and to the right of them, up to and including our pair, and none to the rest of the vertices.

Similarly as in the previous case, we replace the graph H by a graph K that has exactly the arrows listed above (and the same vertices as H). Again, some power of the transition matrix M of K is positive, so we may use the same method for estimating the entropy of K as before. To each black vertex of the i-th pair from the right we assign the component of v equal to 2^i. To the j-th green vertex of K to the left of such a pair (but to the right of the next pair of black vertices) we assign the component of v equal to 2^{i-j}. Now the component of Mv corresponding to a green vertex of K is at most 4 times the component of v corresponding to this vertex. For a black vertex we can bound this ratio by 2 (because of the arrows to black vertices to the right) plus 1 (because of the arrows to the green vertices to the right) plus 2 (because of the arrows to those vertices themselves) plus 4 (because of the arrows to the black vertices of the next pair to the left) plus 1 (because of the arrows to the green vertices to the left). This gives $Mv \leq 10v$. Therefore $h(P) = h(G) \leq h(H) \leq h(K) \leq \log 10$. ∎

From Lemma 5.5 and Proposition 5.6 we get immediately the following theorem.

THEOREM 5.7: *For every $\varrho \in [0, 1/2]$ there exists a sequence of twist patterns with rotation numbers converging to ϱ, entropies bounded from above by* $\log 10$, *and modalities increasing to infinity.*

Appendix: Computation of entropy

The formula we found in Section 3 for $\lambda(\mu)$ looks elegant, but is very impractical for concrete computations. Therefore we will present a "computable" formula for rational values of μ. We denote the integer part of t by $E(t)$.

LEMMA A.1: *If $\mu = p/q$ with p and q coprime, then $\lambda(\mu)$ is the largest root of the polynomial*

$$\sum_{i=0}^{q-1} \varepsilon(i) z^i = 0,$$

where $\varepsilon(i) = -1$ if $i = q-2-E(kq/(q-p))$ for some $k \in \{0,1,\dots,q-p-1\}$ and $\varepsilon(i) = 1$ otherwise.

Proof: We will use some functions defined in [ALMS] and [ALMM] without quoting unnecessary definitions. We are interested only in the dependencies between them and in the definitions of the first and the last functions. All the time we shall use the fact that the largest root of our equation is larger than 1. Therefore we will be able to multiply or divide the equations by polynomials that have no zeros larger than 1.

The first formula from [ALMM] interesting to us is

$$R_{c,d}(z) = \sum_{c<p/q<d} z^{-q}.$$

Thus, our equation for $\lambda(\mu)$ can be rewritten as $R_{\mu,1-\mu}(z) = 1$. Now we will use the following formulas from [ALMS]:

$$Q_{c,d}(z) = (z-1)(1-2R_{c,d}(z)),$$

$$Q_{c,d}(z) = z+1+\frac{2z}{z-1} - 2(T_{1-c}(z) + T_d(z))$$

(we assume that $0 \le c < 1$). Thus, our equation can be rewritten as

$$z+1+\frac{2z}{z-1} - 4T_{1-\mu}(z) = -(z-1),$$

that is

$$2T_{1-\mu}(z) = \frac{z^2}{z-1}.$$

Then we use the formula

$$Q_d(z) = z+1-2T_d(z)$$

from [ALMM], where Q_d is a function from [ALMS]. Our equation becomes simply

$$Q_{1-\mu}(z) = \frac{1}{1-z}.$$

Now we assume that $\mu = p/q$ with p and q coprime. Moreover, we know that $p/q < 1/2$. Lemma 14 of [ALMS] gives us

$$H_{1-\mu}(z) = (z^q - 1)Q_{1-\mu}(z),$$

where

$$H_{1-\mu}(z) = z^{q+1} - z^q - z - 1 - 2\sum_{j=1}^{q-p-1} z^{E(jq/(q-p))+1}.$$

Hence our equation becomes now

$$z^{q+1} - z^q - z - 1 + \sum_{i=0}^{q-1} z^i - 2\sum_{j=1}^{q-p-1} z^{E(jq/(q-p))+1} = 0.$$

Note that $1 < q/(q-p) < 2$, so if $1 \le j \le q-p-1$ then $2 \le E(jp/(q-p)) + 1 \le q-1$. Therefore we can rewrite our equation once more, this time as

$$z^{q-1} - z^{q-2} + \sum_{i=0}^{q-3} z^i - 2\sum_{j=1}^{q-p-1} z^{E(jq/(q-p))-1} = 0.$$

Another way to write it is

$$\sum_{i=0}^{q-1} \varepsilon(i) z^i = 0,$$

where $\varepsilon(q-1) = 1$, $\varepsilon(q-2) = -1$, and for $0 \le i \le q-3$ we have $\varepsilon(i) = -1$ if $i = E(jq/(q-p)) - 1$ for some $j \in \{1, 2, \dots, q-p-1\}$ and $\varepsilon(i) = 1$ otherwise.

Now we notice that if $j \in \{1, 2, \dots, q-p-1\}$ then $jq/(q-p)$ is not an integer. Therefore $E(jq/(q-p)) - 1 = q - 2 - E(kq/(q-p))$, where $k = q-p-j$. If $k = 0$ then $q - 2 - E(kq/(q-p)) = q - 2$, so we get the required formula for $\varepsilon(i)$: $\varepsilon(i) = -1$ if $i = q - 2 - E(kq/(q-p))$ for some $k \in \{0, 1, \dots, q-p-1\}$ and $\varepsilon(i) = 1$ otherwise. ∎

Remark A.2: The formula from Lemma A.1 can be obtained also from the kneading invariant of unimodal twist maps with rotation interval $[\mu, 1/2]$. Alternatively, this kneading invariant can be obtained from our formula. We leave the relevant computations to a curious reader.

We can look at the LRN and the topological entropy in a more global way. For unimodal maps LRN is a coarser invariant than entropy. Indeed, if the kneading invariant of a map is an asterisk product of two kneading invariants, then LRN depends only on the "outer" one. That means that on any set of unimodal maps with the same entropy, LRN is constant. Therefore there is a function $\mu(\cdot)$: $[0, \log 2] \to [0, 1/2]$ such that if $h(f) = \log t$ then $\mathrm{LRN}(f) = \mu(t)$ (cf. Remark 3.10). By Lemma 3.2, this function is non-increasing. Since there are unimodal maps with all lower rotation numbers between 0 and 1/2, this function is also continuous.

It follows that for every μ there is a maximal interval $[\lambda(\mu), \widetilde{\lambda}(\mu)]$ (perhaps degenerated to a point) on which $\mu(\cdot)$ is constant and equal μ. Its left endpoint is the number $\lambda(\mu)$ that we investigated. Its right endpoint is $\widetilde{\lambda}(\mu)$, which is the maximal entropy of a unimodal map with rotation interval $[\mu, 1/2]$. We have $\widetilde{\lambda}(\mu) = \lim_{\nu \nearrow \mu} \lambda(\nu)$, and therefore $\widetilde{\lambda}(\mu)$ is the unique root of the equation $S_\mu(t) = 1$, where

$$S_\mu(t) = \sum_{\mu \le p/q \le 1-\mu} t^{-q}.$$

Therefore $\widetilde{\lambda}(\mu) = \lambda(\mu)$ if μ is irrational and $\widetilde{\lambda}(\mu) > \lambda(\mu)$ if μ is rational.

To obtain an equation for $\widetilde{\lambda}(\mu)$ similar to the one from Lemma A.1, we have to replace in the proof of that lemma $Q_{1-\mu}(z)$ by $\lim_{\nu \nearrow \mu} Q_{1-\mu}(z)$, which is equal to $Q_{1-\mu}(z) - 2(1-z)/(z^q - 1)$ by Lemma 21 of [ALMS]. Therefore the equation $Q_{1-\mu} = 1/(1-z)$ has to be replaced by $Q_{1-\mu} = 1/(1-z) + 2(z-1)/(z^q - 1)$. This leads eventually to the equation

$$\sum_{i=0}^{q-1} \varepsilon(i) z^i = \frac{2(z-1)}{z^2}$$

for $\widetilde{\lambda}(\mu)$.

When we look at the function inverse to $\mu(\cdot)$, we see that it is discontinuous, with $\log \widetilde{\lambda}(\mu) - \log \lambda(\mu) = \log(\widetilde{\lambda}(\mu)/\lambda(\mu))$ representing the jump in topological entropy when the kneading invariant is moving through the values for which the lower rotation number is μ (it occurs only if μ is rational). Our next aim will be to estimate this jump from below and above.

LEMMA A.3: *For every $\mu = p/q \in [0, 1/2]$ with p and q coprime we have*

$$\frac{1}{q} \log \left(1 + \frac{3 - 2\sqrt{2}}{\lambda^q} \right) < \log \frac{\widetilde{\lambda}(\mu)}{\lambda(\mu)} < \log \frac{\lambda^q + 1}{\lambda^q - 1}.$$

Proof: To make the formulas simpler, we will fix $\mu = p/q$ and write $\lambda, \widetilde{\lambda}$ for $\lambda(\mu), \widetilde{\lambda}(\mu)$ respectively.

Assume first that $\mu < 1/2$. Set

$$P(z) = \sum_{i=0}^{q-1} \varepsilon(i) z^i,$$

$$\widetilde{Q}(z) = \sum_{i=0}^{q-1} (z^i - \lambda^i) = \frac{z^q - 1}{z - 1} - \frac{\lambda^q - 1}{\lambda - 1},$$

and

$$\widetilde{R}(z) = \frac{z^q - 1}{\lambda - 1} - \frac{\lambda^q - 1}{\lambda - 1} = \frac{z^q - \lambda^q}{\lambda - 1}.$$

We have

$$P'(z) = \sum_{i=1}^{q-1} \varepsilon(i) i z^{i-1} \le \sum_{i=1}^{q-1} i z^{i-1} = \widetilde{Q}'(z)$$

and $P(\lambda) = \widetilde{Q}(\lambda) = \widetilde{R}(\lambda) = 0$. Therefore $P(z) \le \widetilde{Q}(z)$ for every $z \ge \lambda$. Moreover, for $z \ge \lambda$ we have $z - 1 \ge \lambda - 1$ and $z^q - 1 > 0$ (since $\lambda > 1$), so $\widetilde{Q}(z) \le \widetilde{R}(z)$. Therefore for $z \ge \lambda$ we get $P(z) \le \widetilde{R}(z)$.

Next, by computing the derivative of $2(z-1)/z^2$ we check that this function is increasing in $[\lambda, 2]$. We know that $\lambda < \widetilde{\lambda} < 2$, so

$$\widetilde{R}(\widetilde{\lambda}) \ge P(\widetilde{\lambda}) = \frac{2(\widetilde{\lambda} - 1)}{\widetilde{\lambda}^2} > \frac{2(\lambda - 1)}{\lambda^2} > 0 = \widetilde{R}(\lambda).$$

Therefore there is $\nu \in (\lambda, \widetilde{\lambda})$ such that $\widetilde{R}(\nu) = 2(\lambda - 1)/\lambda^2$. By the definition of $\widetilde{R}$, we get

$$\widetilde{\lambda}^q > \nu^q = \lambda^q + \frac{2(\lambda - 1)^2}{\lambda^2}.$$

Up to now, we were assuming that $\mu < 1/2$. In this case $\lambda \ge \sqrt{2}$, so $2(\lambda - 1)/\lambda^2 \ge 3 - 2\sqrt{2}$, and we get the estimate

$$\log \frac{\widetilde{\lambda}}{\lambda} > \frac{1}{q} \log \left(1 + \frac{3 - 2\sqrt{2}}{\lambda^q} \right).$$

If $\mu = 1/2$ then $\lambda = 1$, $\widetilde{\lambda} = \sqrt{2}$, and $q = 2$, so the above estimate also holds.

We continue with an estimate from above. Again for $\mu < 1/2$, set

$$\widehat{R}(z) = R_{\mu, 1-\mu}\left(\frac{1}{z}\right) = \sum_{\mu < p/q < 1-\mu} z^q, \quad \widehat{S}(z) = S_\mu\left(\frac{1}{z}\right) = \sum_{\mu \le p/q \le 1-\mu} z^q.$$

Then the reciprocals β and $\widetilde{\beta}$ of λ and $\widetilde{\lambda}$ are the smallest positive roots of the equations $\widehat{R}(z) = 1$ and $\widehat{S}(z) = 1$ respectively. Hence $\widetilde{\lambda}/\lambda = \beta/\widetilde{\beta}$. Note also that $\widehat{S}(z) = \widehat{R}(z) + 2z^q/(1 - z^q)$.

Observe that $\widehat{S}$ is convex. Therefore the straight line segment which connects the points $(0, 0)$ and $(\beta, \widehat{S}(\beta))$ lies above the corresponding piece of the graph of $\widehat{S}$. Then $\widetilde{\beta} > \delta$, where $(\delta, 1)$ is the point at which this segment intersects the line $y = 1$. We have $\beta/\delta = \widehat{S}(\beta) = \widehat{R}(\beta) + 2\beta^q/(1 - \beta^q) = (1 + \beta^q)/(1 - \beta^q)$. Therefore $\beta/\widetilde{\beta} < \beta/\delta = (1+\beta^q)/(1-\beta^q)$, so $\widetilde{\lambda}/\lambda < (\lambda^q + 1)/(\lambda^q - 1)$. This gives us an estimate

$$\log \frac{\widetilde{\lambda}}{\lambda} < \log \frac{\lambda^q + 1}{\lambda^q - 1}.$$

If $\mu = 1/2$ then $\lambda = 1$, so the right-hand side of the above inequality is ∞ (not a very useful estimate, but still a valid one). ∎

Remark A.4: We have

$$\log \frac{\lambda^q + 1}{\lambda^q - 1} = \log \left(1 + \frac{2}{\lambda^q - 1}\right) < \frac{2}{\lambda^q - 1}.$$

On the other hand, if q is large then

$$\frac{1}{q} \log \left(1 + \frac{3 - 2\sqrt{2}}{\lambda^q}\right) \sim \frac{3 - 2\sqrt{2}}{q\lambda^q}.$$

Remark A.5: We can modify slightly the estimate from above, to get an estimate of $\log(\nu/\lambda)$ for $\nu \in (\lambda, \widetilde{\lambda}]$ in terms of ν and q. Namely, we use the fact that $\widehat{R}$ is convex and draw a line through $(0, 0)$ and $(\gamma, \widehat{R}(\gamma))$, where $\gamma = 1/\nu$. This line intersects the line $y = 1$ at $(\zeta, 1)$. Then $\zeta > \beta$, so

$$\frac{\nu}{\lambda} = \frac{\beta}{\gamma} < \frac{\zeta}{\gamma} = \frac{1}{\widehat{R}(\gamma)} = \frac{1}{\widehat{S}(\gamma) - \frac{2\gamma^2}{1-\gamma^q}}.$$

Since $\widehat{S}(\gamma) \geq 1$, we get

$$\frac{\nu}{\lambda} < \frac{1 - \gamma^q}{1 - 3\gamma^q} = \frac{\nu^q - 1}{\nu^q - 3}.$$

Therefore

$$\log \frac{\nu}{\lambda} < \log \frac{\nu^q - 1}{\nu^q - 3} = \log \left(1 + \frac{2}{\nu^q - 3}\right) < \frac{2}{\nu^q - 3}.$$

The above estimate is correct only if $\nu^q > 3$. This means that we cannot use it in the case $\mu = 1/2$. We would not do it anyway, since in this case we know that $\lambda = 1$. On the other hand, if $q \geq 4$ then $\nu > \sqrt{2}$, so $\nu^q > 4 > 3$, and the estimate is correct. This leaves the case $q = 3$. Then the only choice for μ is $1/3$, and from Lemma A.1 we get $\lambda = (1+\sqrt{5})/2$. Thus $\nu^q > 2+\sqrt{5} > 3$, so the estimate works also in this case.

We would like to discuss some applications of the above estimates to the conjecture on entropy monotonicity. Computer experiments show that for many one-parameter families of smooth unimodal maps, topological entropy depends on a parameter in a monotone way. However, only for the family of quadratic maps has this monotonicity been proved (see [Ś]). Assume for definiteness that we are looking at a one-parameter family $(f_a)_{0\leq a \leq 1}$ of maps and $h(f_0) = 0$, $h(f_1) = \log 2$. If the entropy grows with the parameter, then $b \geq a$ implies $h(f_b) \geq h(f_a)$. If we cannot prove this, we can still ask: given a, if we know that $b \geq a$, what can we say of $h(f_b)$? Perhaps it is larger than $h(f_a)$ minus some small epsilon?

Here our estimates can help. Namely, there is a large class of families of smooth unimodal maps for which the lower rotation number depends on the parameter in a monotone way. The first set of assumptions guaranteeing this was found in [GT]. Much weaker assumptions (resulting in much wider classes of maps) were found in [B2]. Let us call such a family **LRN-monotone**. The only problem with applying our results is that we want the estimates in terms of $h(f_a)$ rather than λ and q. However, we can cope with this. From our estimates one can easily deduce the following result.

PROPOSITION A.6: *Let (f_a) be an LRN-monotone family of unimodal maps. Fix $q_0 > 2$. Then there are finitely many intervals $[\lambda_j, \widetilde{\lambda}_j]$ such that f_a has rational lower rotation number with denominator smaller than q_0 if and only if $h(f_a)$ is the logarithm of a number from one of the intervals $[\lambda_j, \widetilde{\lambda}_j]$. The numbers λ_j and $\widetilde{\lambda}_j$ are the largest roots of the equations*

$$\sum_{i=0}^{q-1} \varepsilon(i) z^i = 0, \qquad \sum_{i=0}^{q-1} \varepsilon(i) z^i = \frac{2(z-1)}{z^2}$$

respectively, where the coefficients $\varepsilon(i)$ are as in Lemma A.1.

If $h(f_a) = \log \nu$ with $\nu \in [\lambda_k, \widetilde{\lambda}_k]$ then for all $b \geq a$ we have $h(f_b) \geq \log \lambda_k$. If ν does not belong to any of the intervals $[\lambda_j, \widetilde{\lambda}_j]$ then for all $b \geq a$ we have

$$h(f_b) > \max\left(\log \widetilde{\lambda}_k, h(f_a) - \frac{2}{\widetilde{\lambda}_k^{q_0} - 1}, h(f_a) - \frac{2}{\nu^{q_0} - 3}\right),$$

where $\widetilde{\lambda}_k$ *is the largest* $\widetilde{\lambda}_j$ *smaller than* ν.

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